

# The State of Secondary Geometry: A Reflection in Light of the NCTM<sup>1</sup> Standards

By  
Jeffrey O. Bauer  
Wayne State College, Nebraska

## **Preface**

The following narrative will be made available at the author's website: [academic.wsc.edu/mathsci/bauer\\_j](http://academic.wsc.edu/mathsci/bauer_j).

The author recently attended a five-day academy workshop on 9-12 geometry sponsored by the National Council of Teachers of Mathematics (NCTM).

The workshop was very good but the author left it with a feeling that more should have been said and done about geometry education. The curriculum was limited in scope. The curriculum material for the workshop was based on a discussion of the van Hiele model, and on the four NCTM geometry standards as they relate to the 9-12 grades band. The activities were good, but most were obviously intended to be "gimmicky" so as to captivate students' attention.

The author composed the following narrative in an effort to provide "some" of the pieces about geometry education he felt were missing. It is a "State of Secondary Geometry" Address.

## **Introduction**

In *Geometry and the Imagination*, David Hilbert (xxxx) wrote that a "presentation of geometry in large brush-strokes, so to speak, and based on an approach through visual intuition, should contribute to a more just appreciation of mathematics by a wider range of people than just the specialist." Joseph Malkevitch has also echoed similar sentiments in *Geometry's Future* (1991). He later states, "Our students and the public deserve to be more broadly aware of geometric phenomena and applications of geometry" (Malkevitch, 2001).

Hilbert's and Malkevitch's visions for geometry have come at two different times in history (near a century apart), but both share a common theme. The theme describes an intuitive and broad presentation of geometry. Hilbert advocates for a more visual presentation utilizing more realistic models. He is known to have used beer, steins, and tables in place of points, lines, and planes. Such a

---

<sup>1</sup> NCTM (National Council of Teachers of Mathematics)

model could be more easily understood by "today's" students? Malkevitch appears more as an applied geometer and recommends that many topics and concepts be presented to a wider more "lay" audience. He suggests that topics dealing with graph theory, discrete geometry, and convexity be added to the present day geometry curriculum and to show how they are applicable to computer graphics, operation research, robotics, and communications technology.

Geometry and geometry education have a rich history, and much of it has transpired in "modern" times. Modern geometry is considered to consist of the work done since the beginning of the 19<sup>th</sup> century. During this time non-Euclidean geometries have been developed (hyperbolic, elliptic, taxi-cab, etc.). More importantly (to this discussion) there has been a resurgence of interest in Euclidean geometry and the discovery of new theorems (such as Menelaus', Ceva's, Nine-Point Circle, etc.). Another very important event was a more rigorous axiomatizing of Euclidean geometry by Hilbert, Birkhoff, and the School Mathematics Study Group (SMSG).

Geometry education has also experienced reform during the last 50 years. This reform includes the work done by the van Hiele (graduate students/researchers in the Netherlands), the work done by researchers of the former Soviet Union, the work done by the SMSG, the work done by the Consortium for Mathematics and its Applications (COMAP), the efforts of NCTM and recent research done in South Africa.

The development of symbolic algebra and dynamic geometry software has provided both the educator and mathematician with exciting new avenues to explore. The software can also pose difficulties, especially to the educator. How can a solid conceptual base be built using this software?

Let's meander through a brief description and narrative of the information and events just mentioned.

### **van Hiele Model**

The van Hiele model is based on the dissertations written by Dina van Hiele-Geldorf and her husband Pierre van Hiele at the University of Utrecht, Netherlands in 1957 (Crowley, 1987, De Villiers, 1996). Pierre's dissertation attempted to explain why students had difficulty learning geometry. Dina's dissertation was more prescriptive and dealt with the ordering of content and activities. It is from Dina's initial work that the theory begins.

It possesses four main characteristics. These characteristics are summarized as (De Villiers, 1996, Usiskin, 1982):

**fixed order** - The order in which students progress through the thought levels is invariant. In other words, a student cannot be in level  $n$  without having passed the previous level ( $n-1$ ).

**adjacency** - At each level of thought, what was previously *intrinsic*, is now *extrinsic*.

**distinction** - Each level has its own linguistic symbols and own network of relationships connecting those symbols.

**separation** - Two persons who reason at different levels *cannot* understand each other.

The van Hiele's reasoned that the failure of the traditional geometry curriculum resulted from the teacher presenting the subject at a thought level higher than that of the students (De Villiers, 1996). The thought levels are described as (Crowley, 1987, De Villiers, 1996):

**Level 1: Recognition**

Students visually recognize figures by global appearance. They recognize triangles, squares, etc by their shape. Students cannot identify explicit properties of these figures.

**Level 2: Analysis**

Students start to analyze figures and their properties and to learn the appropriate terminology used for describing them. However, students do not interrelate figures or properties of figures.

**Level 3: Ordering**

Students will logically order the properties of figures by short deductive arguments and students will understand relationships between figures (e.g. class inclusions).

**Level 4: Deduction**

Students start to develop longer more complex deductive arguments. Students begin to understand the

importance of deduction, axiomatics, theorems and proofs.

### **Level 5: Integration**

Students review and summarize the learning that has taken place in the previous levels. A synthesis of learned ideas take place but *no new knowledge* is obtained.

High school geometry is usually taught at the third and fourth level, meaning that students must have progressed through the first two levels during grades K-8.

Burger and Shaughnessy (1986) characterized students' thought levels through the first four levels as:

### **Level 1: Recognition**

- Students often use irrelevant visual properties to identify figures, to compare, to classify, and to describe.
- Students usually refer to visual prototypes of figures, and are easily misled by the orientation of figures.
- Students show an inability to think of an infinite variation of a particular type of figures.
- Students use inconsistent classifications of shapes; they use obscure or irrelevant properties to classify figures.
- Students provide incomplete descriptions (definitions) for shapes by using necessary (often visual) but not sufficient conditions.

### **Level 2: Analysis**

- Students explicitly compare figures based on underlying properties.
- Classes of figures still remain disjoint, e.g. a square is not a rectangle.
- Students sort figures in terms of one property (usually a simpler property, e.g. use of sides over symmetric relationships).
- Students use very lengthy definitions that are not "economical".
- Students tend to reject others' definitions even when they come from a more authoritative source.
- Students will make more empirical arguments through use of observation and measurement.

**Level 3: Ordering**

- Students begin to formulate correct economical definitions of concepts.
- Students exhibit an ability to complete incomplete definitions and students show a willingness to accept definitions for new concepts.
- Students will accept different definitions for the same concept.
- Students hierarchically classify figures.
- Students use the logical "if ... then" form to form and handle conjectures. They also implicitly use rules of logic.
- Students are uncertain and unclear about axiomatics.

**Level 4: Deduction**

- Students are aware of the role of axiomatics.
- Students will spontaneously conjecture and initiate efforts to deductively verify conjectures.

**Russian Studies**

Geometry has always played a key role in Russian mathematics, probably due to the great Russian geometers (like Lobachevsky) and the great psychologists (such as Pavlov) (Kilpatrick & Wirzup, 1969, De Villiers, 1996). The traditional Russian geometry curriculum is split into an *intuitive* phase and a *deductive* phase. The deductive phase usually starts at the sixth grade.

During the late 1960s Russian researchers analyzed the geometry curriculum in efforts to determine why students that exhibit progress in other subjects do not exhibit progress in geometry. The van Hiele model served as the basis of much of their analysis. They discovered that by the end of the intuitive phase, only 10-15% of the students were at the second thought level (to start the deductive phase of their curriculum students need to be at least at the third van Hiele level) (De Villiers, 1996).

The main reason for student difficulties was attributed to inadequate preparation in elementary school. Another problem that became obvious was the mismatch in learning level and teaching level. Teachers would teach at the third van Hiele level or higher. The better teachers would try to present material at level 1, level 2 and level 3, concurrently. This could very easily turn into a juggling act in which the balls are dropped.

Two analyses performed by V.I. Zykova (Kilpatrick & Wirzup, 1969) concentrated on the use of visual aides (drawings) and how students were able to interpret the drawings. Students showed two main problems: (1) the students could not establish connections between related concepts, and (2) the scope of the student learned concepts were limited to figures in "standard" position. The problem of students not mastering concepts "stems from the instructional process" (Kilpatrick & Wirzup, 1969, p 145).

How can student mastery of geometric concepts be obtained? It "depends considerably on how well the teacher coordinates his explanations with the geometric visual aides available" (Kilpatrick & Wirzup, 1969, p 183).

The Russians developed a very successful geometry curriculum based on such analyses and founded on the van Hiele model. An important factor to the success of the curriculum is a continuous sequencing and development of geometric concepts in elementary school (De Villiers, 1996). Wirzup (1976) reported that the average eighth grader of the new curriculum showed the same or better understanding than did an eleventh or twelfth grader in the previous curriculum.

### **The School Mathematics Study Group**

The School Mathematics Study Group (SMSG) was founded by the National Science Foundation (NSF) in 1958, midst the "Cold War" as a response to perceived notions that the United States were falling behind the Soviet Union in basic mathematics research and education (Wallace & West, 1998). Edward G. Begle (19xx-19xx) of Yale University directed the group. The group developed several high school mathematics textbooks pertaining to several subjects, and field-tested them in the early 1960s. Although several of the textbooks were highly criticized, the geometry texts received positive reviews and various forms of the SMSG axiom set have been used in secondary schools in the U.S. and Canada.

### **Modern Axiomatics**

During the 19<sup>th</sup> century it became obvious that there was no single axiom set that would produce a universal model of geometry. Great progress was made in development of the non-Euclidean geometries (hyperbolic and elliptic). It was also evident that no single axiom set could be found for Euclidean geometry. However, mathematicians agreed that the axiom set must be consistent, independent, and complete.

Three significant axiom sets were developed during the 19<sup>th</sup> and 20<sup>th</sup> centuries that result in the same theorems of Euclid's (xxx B.C.) *Elements*.

One such set was developed by the German geometer David Hilbert (1862-1943). Hilbert's axiom set was first published in 1899 as "*Grundlagen der Geometrie (Foundations of Geometry)*", which is true to the spirit within which Euclid worked" (Wallace & West, 1998, p. 45). Hilbert, being familiar with modern axiomatics, avoided some of the difficulties of Euclid and clearly established the unknown terms as *point*, *line*, *plane*, *on*, *between*, and *congruence*. Hilbert's approach remained synthetic like Euclid's.

A second axiom set developed, was by George D. Birkhoff (1884-1944) during the end of his career. The set came near 50 years after Hilbert's. Birkhoff's axioms place measurement at their center. The set is considerably smaller than Hilbert's and establishes *point*, *line*, *distance*, and *angle* as undefined terms. Birkhoff's approach is considered to be analytic in nature.

"The streamlined nature of Birkhoff's axiom set has made it attractive to many mathematicians. The ease with which one can address the issues of betweenness, congruence, and similarity (among other topics) made this approach pedagogically preferable to Hilbert's in many ways" (Wallace & West, 1998, p. 57). As a result, Birkhoff's set appears in many high school geometry texts, allowing for a " 'rigorous' but not cumbersome, discussion of many standard topics in geometry" (Wallace & West, 1998, p. 57).

The SMSG formulated an axiom set during the 1960s that includes the ruler and protractor postulates. But, the set is not independent like Hilbert's and Birkhoff's sets. Theorems are included in the set.

**The Ruler Postulate.** The points of a line can be put into one-to-one correspondence with the real numbers in such a way that (1) to every point there corresponds exactly one real number, (2) to every real number there corresponds exactly one point of the line, and (3) the distance between two points is the absolute value of the difference of the corresponding numbers (Wallace & West, 1998, p. 60).

**The Angle Measure Postulate.** To every angle  $\angle ABC$  there corresponds a unique real number between 0 and 180 (Wallace & West, 1998, p. 61).

**NCTM's Efforts**

During the 1970s the mathematics communities started to notice and incorporate the van Hiele model into their geometry curriculum. However, the 1973 NCTM yearbook, *Geometry in the Mathematics Curriculum*, did not give a detailed explanation of the van Hiele model. The 1987 NCTM yearbook, *Learning and Teaching Geometry, K-12*, devoted its first chapter to the model.

In 1989 the NCTM published its landmark document, *Curriculum and Evaluation Standards for School Mathematics*. In this document, Standard 7 refers to the study of geometry synthetically, whereas Standard 8 refers to the study of geometry algebraically. The *Curriculum and Evaluation Standards for School Mathematics* document also advocates for the inclusion of synthetic, coordinate, and transformational geometry in the high school mathematics curriculum. This document has also been interpreted as down playing the importance of proof in the high school development of mathematics and geometry. J. Michael Shaughnessy and William F. Burger (1985) are also known to support this notion. They are credited with the statement: "Secondary school students should study geometry without proof for at least one-half year".

In 2000 the NCTM released its publication, *The Principles and Standards for School Mathematics* (referred to as *The Principles*). This document also has a recently released companion document known as *Navigating Through Geometry* (2001). This document is divided into four volumes corresponding to the four grades bands described in *The Principles*.

Both documents include geometry as one of its major areas of study in all grades bands. The focus of geometry is an emphasis on a student's ability to:

- analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships;
- specify locations and describe spatial relationships using coordinate geometry and other representational systems;
- apply transformations and use symmetry to analyze mathematical situations; and
- use visualization, spatial reasoning, and geometric modeling to solve problems (NCTM, 2000, p. 41).

Both documents also discuss the benefits of proof. "By the time students reach high school, they should be able to extend and apply geometric knowledge developed earlier to establish or refute conjectures, deduce new knowledge from previously established facts, and solve geometric problems" (NCTM, 2001, p. 4). The NCTM goes on to say that as students mature in their geometric ability, they are to "understand the role of definitions and axioms and to appreciate the connectedness of logical chains, recognizing, for example, that if a result is proved true for an arbitrary parallelogram, then it automatically applies to all rectangles and rhombuses" (p 4). This is what De Villiers (1996) refers to as the use of hierarchical definitions.

The *Navigating Through Geometry* series provides educators with explanation of the four geometry standards, as well as activities deemed appropriate to the standards. The individual books are divided into four chapters corresponding to the four standards. There is also an appendix containing black-line masters to go with the described activities.

#### **COMAP: Geometry's Future**

In 1990, COMAP sponsored a workshop for educators and researchers to discuss the importance of teaching geometry and its apparent stagnation. The outcome was the publication of a document entitled *Geometry's Future*. Joseph Malkevitch, of York College (CUNY) edited *Geometry's Future* (1991). The document includes the following twelve recommendations:

- Geometric objects and concepts should be more studied from an experimental and inductive point of view rather than from an axiomatic point of view. (Results suggested by inductive approaches should be proved.)
- Combinatorial, topological, analytical, and computational aspects of geometry should be given equal footing with metric ideas.
- The broad applicability of geometry should be demonstrated: applications to business (linear programming and graph theory), to biology (knots and dynamical systems), to robotics (computational geometry and convexity), etc.
- A wide variety of computer environments should be explored (Mathematica, LOGO, etc.) both as exploratory tools and for concept development.

- Recent developments in geometry should be included. (Geometry did not die with either Euclid or Bolyai and Lobachevsky.)
- The cross-fertilization of geometry with other parts of mathematics should be developed.
- The rich history of geometry and its practitioners should be shown. (Many of the greatest mathematicians of all time: Archimedes, Newton, Euler, Gauss, Poincaré, Hilbert, Von Neumann, etc., have made significant contributions to geometry.)
- Both the depth and breadth of geometry should be treated. (Example: Knot theory, a part of geometry rarely discussed in either high school or survey geometry courses, connects with ideas in analysis, topology, algebra, etc., and is finding applications in biology and physics.)
- More use of diagrams and physical models as aids to conceptual development in geometry should be explored.
- Group learning methods, writing assignments, and projects should become an integral part of the format in which geometry is taught.
- More emphasis should be placed on central conceptual aspects of geometry, such as geometric transformations and their effects on point sets, distance concepts, surface concepts, etc.
- Mathematics departments should encourage prospective teachers to be exposed to both the depth and breadth of geometry (Lee, 1998).

The document made a distinction between the mathematics of geometry and actual physical geometry. Students' ability to understand proofs and their seeing examples of mathematical proofs was deemed very important.

Joseph Malkevitch continues to advocate for a reform in geometry and geometry education that he refers to as *Geometry in Utopia* (first described in *Geometry's Future*) (Malkevitch, 2002). He believes that geometry is "significantly under represented in the curriculum." He also feels that this "has come about because mathematicians and the public identify geometry with the axiomatic geometry taught in high school years ago. Geometry, in fact, is a vast subject covering a wide variety of different arenas" (Malkevitch, 2002). Traditional topics in geometry require augmentation with ideas from theory of graphs, discrete geometry, and convexity to show a full

range of tools that are available for solving problems (Malkevitch, 1999). These geometric tools are driven by and applicable to computer graphics, operation research, robotics, and communications technology, as well as many other areas. A global presentation of geometry is what he refers to as *Geometry in Utopia*.

It is hard to imagine all of the concepts and topics of Malkevitch's *Geometry in Utopia* being left until the secondary school years. According to Malkevitch (2001) students leaving the sixth grade have seen very little geometry, though it would be reasonable for them to have seen many of the geometric concepts from *Geometry in Utopia*. Geometry is built upon many "parts" that overlap. Several of these parts are accessible by students who are at the first van Hiele level. Thus, elementary school children can be exposed early.

### **South Africa**

According to Michael De Villiers (1996) South Africa's experience with its geometry curriculum is very similar to the Russian experience. The geometry curriculum is very heavily loaded in the secondary years with a formal geometry course. Very little is done in the primary grades and when concepts are presented it is done informally.

The van Hiele model appears to provide the answers to the cause of South African students difficulty with geometry. Research done by De Villiers and Njisane (1987) indicated that a great portion of students had only mastered van Hiele level 2 or lower by twelfth grade. These same results were found in studies done by Malan (1986), Smith and De Villiers (1990), and Govender (1995) (cited in De Villiers, 1996).

These same studies also depicted that second language students had difficulty with the transition from Level 1 to Level 2. The students had to learn the technical terminology used to describe geometric figures.

"It seems clear that no amount of effort and fancy teaching methods at the secondary school will be successful, unless we embark on a major revision of the primary school geometry curriculum along van Hiele lines" (De Villiers, 1996). The state of secondary geometry ultimately depends on the state of elementary geometry.

Japanese students experience planar and spatial investigations as early as first grade. These investigations are continually built upon, so by the time fifth grade is reached, students are formally dealing with similarity and congruence.

South African schools recently introduced tessellations into their elementary school curriculum. According to De Villiers (1996), triangular tessellations can be used to provide an intuitive visual foundation (van Hiele Level 1) for a variety of geometry, which can later be investigated deductively.

For example De Villiers (1996) suggests that the tessellation illustrated below (Figure 1) and the following questions constitute a Level 1 van Hiele episode. Students would be asked:

1. Identify and color parallel lines.
2. What can be said about angles  $\angle A$ ,  $\angle B$ ,  $\angle C$ ,  $\angle D$  and  $\angle E$  and why?
3. What can be said about  $\angle A$ ,  $\angle 1$ ,  $\angle 2$ ,  $\angle 3$  and  $\angle 4$  and why?

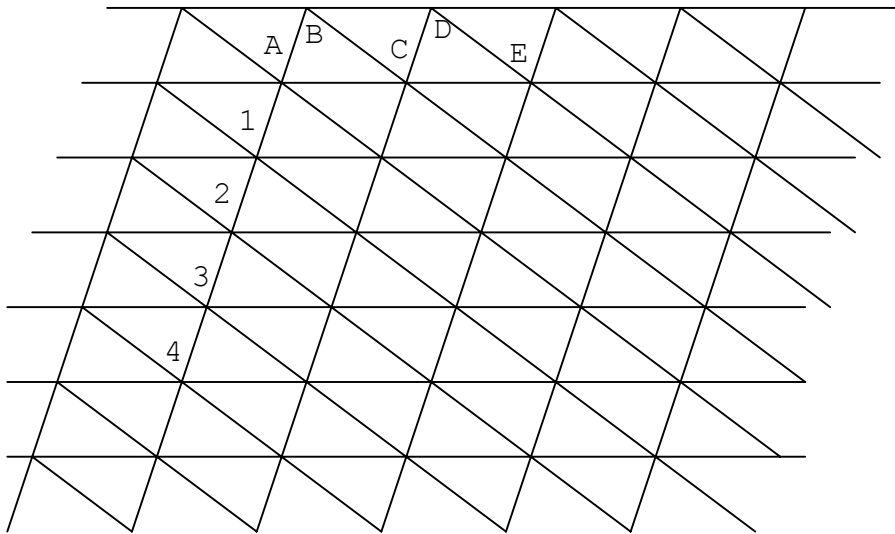


Figure 1: van Hiele Level 1 (Visualization)

The same tessellation could also be used to create a Level 2 episode (De Villiers, 1996). Students would be asked:

1. What can be said about  $\angle A$  and  $\angle B$  in relation to  $\angle D$  and  $\angle E$ ? Why? What conclusion can be made?
2. What can be said about  $\angle F$  and  $\angle G$  in relation to  $\angle H$  and  $\angle I$ ? Why? What conclusion can be made?
3. What can be said about segment  $\overline{JK}$  in relation to segment  $\overline{LM}$ ? Why? What conclusion can be made?

A Level 3 van Hiele episode occurs when students argue that;  $\angle 1$  is congruent to  $\angle 2$  by a SAW,  $\angle 1$  is congruent to  $\angle 3$  by a LADDER, therefore  $\angle 2$  is congruent to  $\angle 3$ .

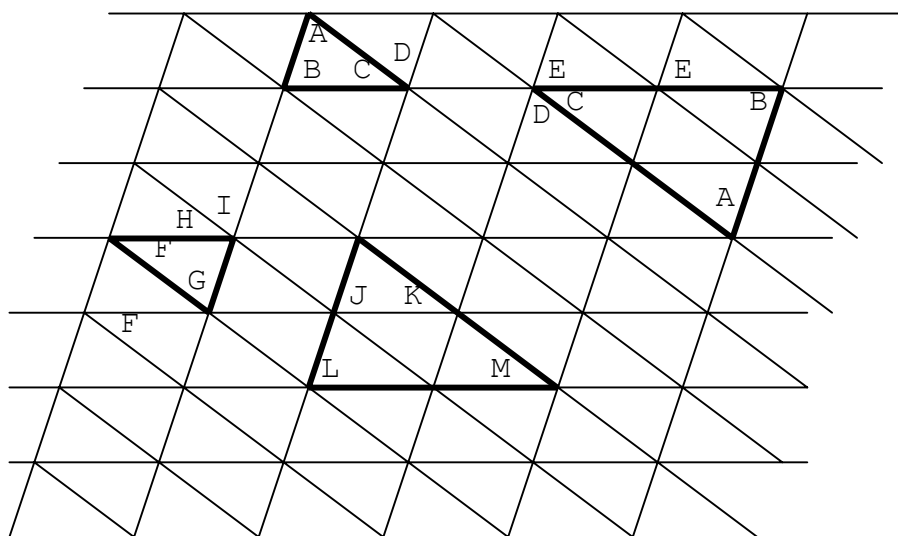


Figure 2: van Hiele Level 2 (Analyzing)

A very important aspect of a van Hiele model is that informal activities at Level 1 and Level 2 should provide a conceptual foundation for the next level. Having students measure the angles of several triangles to verify that the sum of the angles of a triangle is  $180^\circ$ , does not qualify as a good van Hiele episode. The activity does not allow for growth or transition to a next van Hiele level. A better alternative would be to use the tessellation above and observe the rotations and translations of the triangles to show the sum to be  $180^\circ$ .

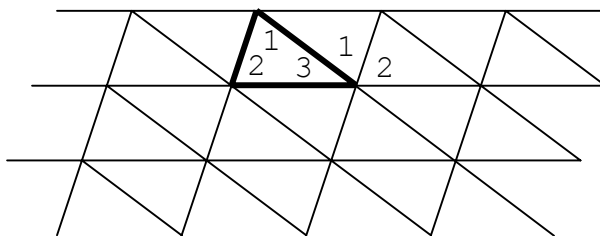


Figure 3: Sum of the Angles of a Triangle is  $180^\circ$ .

The observations above can also lead to the development of a formal proof by students (van Hiele Levels 3 & 4).

### Process versus Product

Traditional geometry education is comparable to a cooking class where the teacher doesn't let you cook (De Villiers, 1996). What is even worse, the teacher shows you pictures of food without telling you what the ingredients are.

The distinctions made by De Villiers (1996) between processes and products of formal geometry are shown in the table below (Table 1).

Most products have several processes associated with them, for example, notice the processes associated with theorems in Table 1. In similar fashion, each process has a product associated with it. The process of proving has a product, a proof. A proof is distinguishable from axioms, definitions and theorems, even though these three products are used in proofs.

Product	Process
Axioms	Axiomatizing <ul style="list-style-type: none"> <li>• Proving</li> </ul>
Definitions	Defining <ul style="list-style-type: none"> <li>• Experimenting</li> <li>• Proving</li> </ul>
Algorithms	Algorithm construction and verification
Theorems	Theorem finding and formulating <ul style="list-style-type: none"> <li>• Experimenting</li> <li>• Refuting</li> <li>• Pattern finding</li> <li>• Generalizing</li> <li>• Specializing</li> <li>• Visualizing</li> <li>• Proving</li> </ul>
Classifications	Classifying

Table 1: Product vs. Process (De Villiers, 1996)

Another product of formal geometry is the definition. A formal definition is a product of Level 3 of the van Hiele model. Previous discussion should show the introduction of formal definitions prematurely (in say Level 1 or Level 2) would be an exercise in futility.

Taking a constructivist approach, definitions can be handled in the van Hiele model. The concept of visual definitions can be introduced at Level 1 and "uneconomical" definitions can be introduced at Level 2. A visual definition would mean that a figure is identified by basic attributes, e.g. a rectangle is a quadrilateral with all right angles and two long and two short sides. An uneconomical definition for a rectangle would be; a rectangle is a quadrilateral with opposite sides parallel and equal, all right angles, equal diagonals, possesses half-turn symmetry, two axes of symmetry, two long and two short sides, etc. The correct formal (economical) Level 3 definition would read as: a rectangle is a quadrilateral with two axes of symmetry through opposite sides.

The definition of Level 2 is also considered *partitional*, i.e. the definition above would not include a square as a member of the class of rectangles. Level 3 definitions are considered *hierarchical*, i.e. a definition for a parallelogram would include a square, rectangle, and rhombus.

A true constructivist educator would not introduce students to ready-made definitions until they had passed through Levels 1 and 2. Once in Level 3 students are able to accept different definitions for the same concepts (Burger & Shaughnessy, 1986).

### **The USEME<sup>2</sup> Experiment**

The USEME experiment was conducted with a control group and an experimental group (Human & Nel et al, 1989a: cited in De Villiers, 1996). The experiment itself was aimed at tenth graders. The control group received a traditional instructional approach whose end product was deductive proofs. The experimental group's instruction was aimed at developing the ability of constructing formal economical definitions for geometric concepts, and developing student understanding of the nature and role of axioms, definitions and proofs. The approach concentrated more on process.

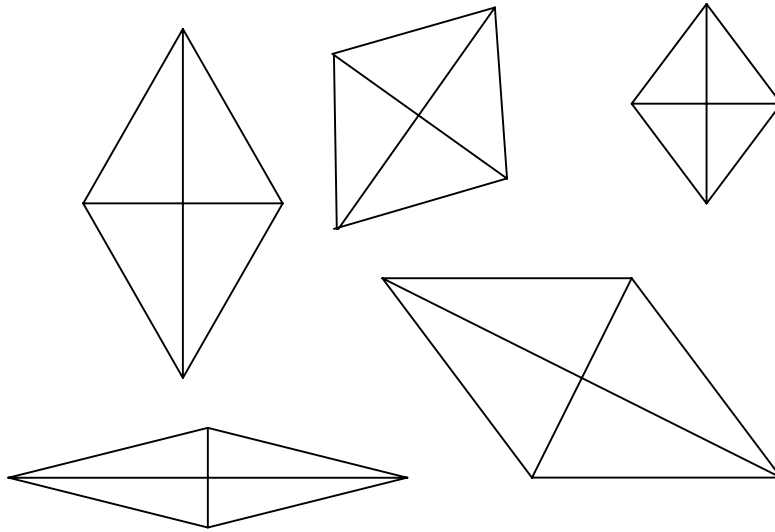
The following is an example of one of the exercises used in the experimental approach (Human & Nel et al, 1989b: cited in De Villiers, 1996).

#### **Exercise**

- 1(a) Make a list of all the common properties of the figures below. Look at angles, sides, and diagonals and measure if necessary.

---

<sup>2</sup> USEME (University of Stellenbosch Experiment with Mathematics Education)



- (b) What are these types of quadrilaterals called?
- (c) How would you explain in words, *without making a sketch*, what these quadrilaterals are to someone not yet acquainted with them?

This exercise led students to the creation of partitional definitions (Level 2). Students were brought into Level 3 by additional exercises such as:

**Exercises**

2 A letter is addressed as follows:

Mr. JH Nel  
 Nelsterede  
 9 Ventor Avenue  
 PO Box 48639  
 Stellenbosch  
 7600

- (a) The address is unnecessarily long. Give a shortened version of the above address so that Mr. Nel will still receive the letter.
  - (b) Are there other shortened versions of the above address whereby the letter would still reach Mr. Nel?
- 3(a) Construct three different rhombi on your own.  
 (b) Refer back to your verbal description of rhombi in 1(c). Is your description unnecessarily long? Can you give a shorter description that will

- still produce a rhombus if you attempted to construct it from the description?
- (c) Give three different shorter descriptions of a rhombus.
  - (d) Try constructing a quadrilateral that is not a rhombus but still complies with your first shorter definition. Check your other shorter descriptions in the same manner.

Psychologically, constructions like these are important to make the transition from van Hiele Level 2 to Level 3 (De Villiers, 1996). It helps to develop an understanding of the difference between a premise and conclusion and reinforces their relationship. It highlights the logical structure of the "if ... then" construct. According to Smith (1940) students make marked improvement in understanding this construct if they make constructions to verify the validity of geometric statements.

Students in the experimental group were then led into a deductive phase. Students were given a definition of a rhombus and asked to logically check if the other properties they had observed could be derived (Human & Nel et al, 1989a). To avoid confusion, it was later agreed to select only one definition for a rhombus to be used by everyone. A parallelogram was treated in similar fashion.

The USEME project tried to show the students the necessity of definitions and axioms in a geometric system. To do this, exercises that forced students to make circular arguments, were investigated. The gist of one of the exercises was for students to prove that if two parallel lines were cut by a transversal, then alternate angles are equal. The proof was to be based on the assumptions that corresponding angles are congruent, that co-interior angles are supplementary, and on the vertical angle theorem. Students were then asked to prove any assumptions that they used along the way. Circular arguments were soon found.

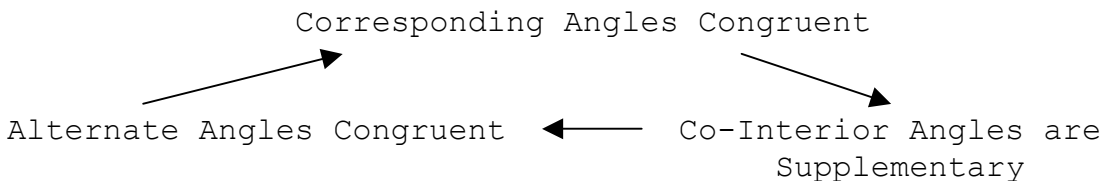


Figure 4: Circular Argument

Comparative research after the experiment showed that participants in the experimental group significantly outperformed the control group in their ability to create

hierarchical definitions, and showed substantially greater and deeper understandings of axiomatics.

### **NCTM Academy**

The NCTM sponsors workshops based on its recently released *Principles*. The workshops are known as Academies and currently deal with algebra and geometry. Parallel with instituting these workshops was the publication of the *Navigating Through Algebra*, and *Navigating Through Geometry* series.

The author recently attended a five-day Academy on 9-12 geometry. The curriculum for the workshop was organized around four components directly related to the four geometry standards in the *Principles* and a culminating activity.

- Analyze properties of geometric shapes.
- Specify locations and describe spatial relationships using coordinate geometry and other representations.
- Analyze transformations and use symmetry to analyze mathematical situations.
- Use visualization, spatial reasoning, and geometric modeling to solve problems.
- Culminating exercise; Fractals (Geometric Probability?).

The workshop started out with a very limited discussion of the van Hiele model for geometry education. It appeared to be the consensus of the participants that most of the students that they dealt with in high school are stuck at van Hiele Level 2.

A day was devoted to each component. The objective for the first day was: Analyze properties of geometric shapes. This objective was broken into the following six sub-goals (activities):

- Quadrilateral sort.
- Mid-segment investigation.
- Mid-segment investigation with dynamic geometric software.
- Proof of mid-segment theorem.
- Prove Varignon's theorem.
- Exploring 3-D figures.

The quadrilateral sort activity's goal was to *analyze characteristics and properties of 2-D geometric shapes and develop arguments about geometric relationships*. The activity page is from *Navigating Through Geometry in Grades 3-5*. There is a similar activity in the NCTM Addenda Series, *Geometry in the Middle Grades*, page 58. The activity involved a sorting of various quadrilaterals by common characteristics. The activity's dynamics were changed when individuals were asked to locate a single quadrilateral possessing a characteristic unique from the others, then asked to find two quadrilaterals that shared a common trait, etc. The activity, in the author's opinion, should have a van Hiele Level 2 classification (as it was administered).

The second activity dealt with the mid-segment theorem. The activity started by constructing the mid-segment using available tools such as straightedge and compass, a reflecting mirror, tissue paper, etc. The segment could also be constructed using dynamic software such as *Geometer's Sketchpad* or *Cabri'*. You were then asked to look for the characteristics of the segment. Only after a list of characteristics was given and conjectures made, was the theorem stated.

**Mid-Segment Theorem.** In a triangle the segment joining the midpoints of two adjacent sides, is both parallel to the third side and half its length. This segment is known as the mid-segment.

Individuals were then asked to develop a proof of this theorem. It was hoped that sample proofs would be indicative of the more traditional narrative synthetic style, the coordinate or vector style, or a transformational style. Most participants used the more traditional style and some used a coordinate style after some coaxing.

The mid-segment theorem was then used as a springboard into Varignon's theorem.

**Varignon's Theorem.** In a quadrilateral the segments joining the midpoints of two adjacent sides form a rectangle.

This investigation was done in a manner similar to the one used for the mid-segment theorem. These activities, based on the author's thoughts, deal with concepts in the third and fourth van Hiele level.

The last activity of the first day dealt with the characteristics and properties of the Platonic solids: (1) tetrahedron, (2) cube, (3) octahedron, (4) dodecahedron, and (5) icosahedron. The characteristics sought were the number of vertices, the number of sides, and the number of faces for each of the five solids (Table 2). These characteristics were discovered through the use of snap-together manipulatives (constructions).

The presenters were hoping that Euler's formula would be discovered by this exercise.

**Euler's Formula.**  $F-E+V=2$ , where  $F$  is the number of faces,  $E$  is the number of edges, and  $V$  is the number of vertices.

It was also hoped that participants would discover the reason that there exist only five regular polyhedra. This discovery was to be accomplished through the use of the takeout angle. The takeout angle is said to be the angle remaining from  $360^\circ$  once the faces that meet at a common vertex are tiled in a plane. For a tetrahedron three equilateral triangles meet at a common vertex. This accounts for  $60^\circ + 60^\circ + 60^\circ = 180^\circ$  leaving  $180^\circ$  (refer to Figure 5). Since the total angle measure is less than  $360^\circ$ , the shape can be folded into 3-space to form a hat. This means that a Platonic solid can be formed. A final observation made was that the number of vertices,  $V$ , times the takeout angle,  $T$ , is  $720^\circ$  ( $V \times T = 720^\circ$ ).

Polyhedra	# Vertices	# Edges	# Faces	Takeout Angle
[3, 3, 3] Tetrahedron	4	6	4	$180^\circ$
[3, 3, 3, 3] Octahedron	6	12	8	$120^\circ$
[3, 3, 3, 3, 3] Icosahedron	12	30	20	$60^\circ$
[4, 4, 4] Cube	8	12	6	$90^\circ$
[5, 5, 5] Dodecahedron	20	30	12	$36^\circ$

Table 2: Polyhedra Investigation

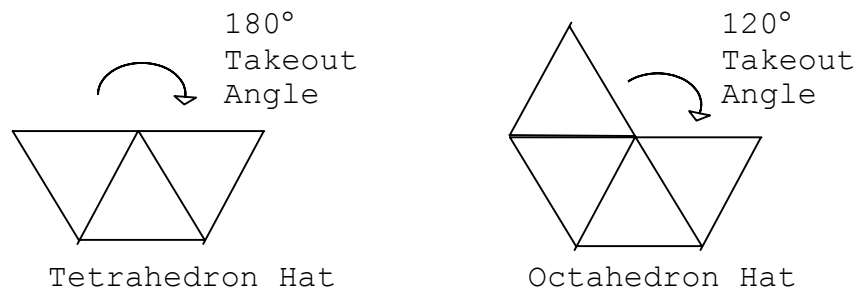


Figure 5: Nets for Hats to Platonic Solids

A more complete description of the rest of the workshop can be obtained from the author later or you may consider enrolling in one of the NCTM Academy workshop.

The objective for the second day was: *Specify locations and describe spatial relationships using coordinate geometry and other representations.* The associated activities for the day included:

- Real-life examples of, points, segments, lines, and planes.
- Delivering packages.
- Where are we now?
- Intuitive cartography.
- Anamorphic art.
- Fermat's point.

The objective for the third day was: *Analyze transformations and use symmetry to analyze mathematical situations.* The associated activities for the day included:

- Reflections over the line  $y = x$  on the coordinate plane.
- Composition of reflections.
- Pass the protractor (linear regression).
- Gliding along.
- Basic dilations.
- Computer graphics on a graphing calculator (TI-83).

The objective for the fourth day was: *Use visualization, spatial reasoning, and geometric modeling to solve problems.* The associated activities for the day included:

- Introduction to graph theory.
- Graph theory.
- Least squares investigation.
- Polyhedron slicing.

The objective for the last day was to summarize the week and to look at a final concept, fractals.

The workshop was for the most part very good. It gave the author a chance to visit with secondary teachers from several different states. These visits gave him insight into several of the concerns and practices present in geometry education today. The material of the workshop addressed several of these concerns and/or practices. However, at times the presentation drifted from the van Hiele model and some of the activities were "gimmicky". But, gimmicky does fit with Malkevitch's *Geometry in Utopia* approach to content. The most common questions asked by the participants dealt with curriculum and most *overwhelmingly with technology*.

### **Dynamic Geometry Software**

The development of geometry software is very exciting and according to De Villiers (1996) it is responsible for saving Euclidean geometry from the trash heap in many countries. This type of software has also helped with student difficulties along the van Hiele lines. Studies already cited have indicated that one great difficulty is the ability to visualize on the students' part. Dynamic software can help to remedy this problem.

Dynamic software can also alleviate the tedious nature of constructing several different examples by both teacher and student. Thus, giving each more time to explore and investigate other concepts.

Michael De Villiers (1996) feels that the most welcomed facility to dynamic software is its potential to encourage student experimentation and "research" as described by Luthuli (1996) and others. Such a research approach to geometry education inducts the students into a thought process of exploration, conjecturing, refuting, reformulating, and explaining (Figure 6).

Dynamic software has changed the traditional approach of trying to instill doubt as a means of motivating students to make deductive arguments, to an approach of this is happening, why? Deductive arguments are now seen as explanations rather than as verifications.

The following is a possible activity for dynamic software (De Villiers, 1995):

**Worksheet**

- (a) Construct a dynamic kite using the properties of kites explored and discussed in our previous lessons.
- (b) Check to ensure that you have a dynamic kite, i.e. does it always remain a kite no matter how you transform the figure? Compare your construction(s) with those of your neighbors - is it the same or different?
- (c) Next construct the midpoints of the sides and connect the midpoints of adjacent sides to form an inscribed quadrilateral.
- (d) What do you notice about the inscribed quadrilateral formed in this way? (Make some measurements to check your observation).
- (e) State your conjecture.
- (f) Grab any vertex of your kite and drag it to a new position. Does it confirm your conjecture? If not, can you modify your conjecture?
- (g) Repeat the previous steps a number of times.
- (h) Is your conjecture also true when your kite is concave?
- (i) Use the property checker of *Cabri'* to check whether your conjecture is true in general.
- (j) State your final conclusion. Compare with your neighbors - is it the same or different?
- (k) Can you explain **why** it is true? (Try to explain it in terms of other well-known geometric results. *Hint*: construct the diagonals of your kite. What do you notice?)
- (l) Compare your explanation(s) with those of your neighbors. Do you agree or disagree with their explanations? Why? Which explanations are the most satisfactory? Why?

(Note: This exercise illustrates a specific case of Varignon's theorem.)

Proof has many functions, e.g. verification, systemization, communication, discovery, intellectual challenge, etc. Dynamic software allows students to explore conjectures for many different cases. Proof becomes an explanation. The teacher must tie the other functions of proofs together with the explanation function in a "spiral" approach. The explanation role can be introduced as early

as van Hiele Level 1, whereas the systemization role must be postponed until students are at Level 4.

Students at van Hiele Level 1 would have difficulty with dynamic constructions. However, they would do quite well with readymade constructions. Dynamic constructions will also make hierarchical definitions more believable.

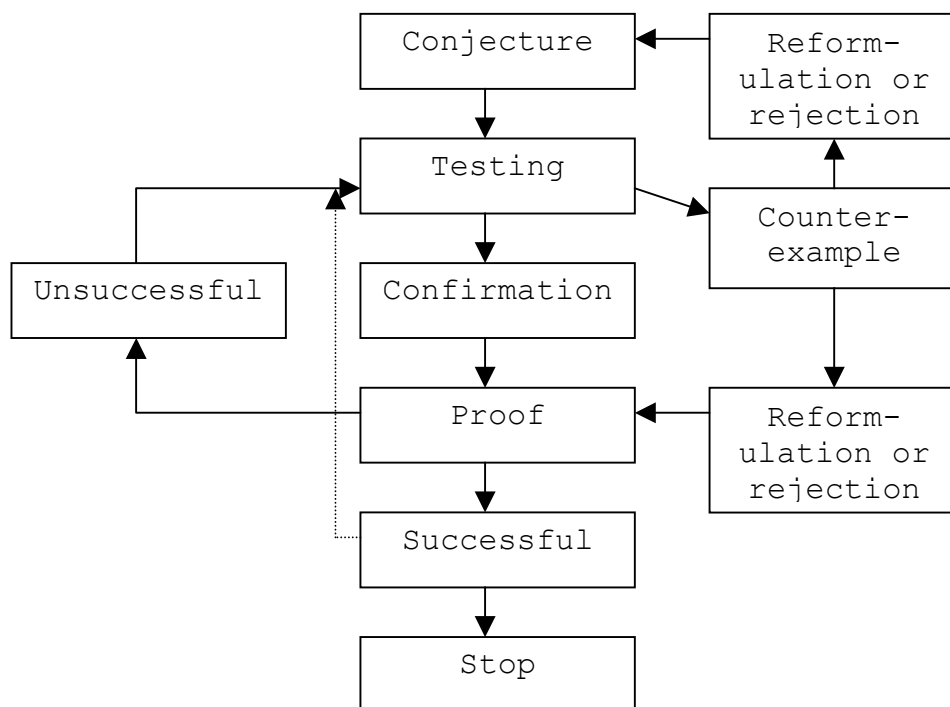


Figure 6: Student Research in Geometry (De Villiers, 1996).

### Concluding Remarks

The author was first inspired to write this narrative after attending a NCTM Academy workshop on 9-12 geometry. He felt that there was much more to be said about geometry and geometry education than was presented. However, **the author does support the NCTM and agrees with its standards for geometry. He just feels that the standards need to go farther.**

The van Hiele model remains a valid psychological model for the instruction and learning of geometry. Emphasis should also be given to the individual learning style within each of the van Hiele levels. Gregorc's classifications could be used to provide this emphasis. The Gregorc classifications are: (1) concrete sequential, (2) abstract sequential, (3) concrete random, and (4) abstract random.

The curriculum content should definitely include the NCTM recommendations. **Students should be exposed to all sorts of applications and opportunities to solve realistic problems.** These opportunities should lead students to analyze the characteristics of both 2-D and 3-D objects and logically argue significant relationships. Students should also be exposed to different coordinate systems and spatial relationships. The very definition of geometry, tells us that students should be able to understand and apply transformations.

But, the author feels that axiomatics and the process of proving remain very important. It is a concept of the third and fourth van Hiele levels, which students typically do not reach until high school. Therefore, axiomatics and the process of proving should be relegated to the high school mathematics curriculum. Studies have shown that if the creation of definitions is center stage during the students' transition through the first three van Hiele levels, students will not only excel at definitions but also at the process of proving. If quality instructional time is given to the process of proving, students should then be able to understand and produce proofs. As a by-product, students should also be able to understand definitions, axioms, and theorems (important products identified by De Villiers). **It is to be understood that axiomatics and proving should not be done solely in a formal geometry class, but done throughout the entire high school mathematics curriculum.**

The elementary geometry curriculum, according to the NCTM, should concentrate on the same four standards, as does the secondary curriculum. The previous discussion should also have made it quite evident that without a strong elementary mathematics curriculum that concentrates on geometry and on transitioning students through the first two van Hiele levels, the efforts of the secondary geometry teacher are doomed. This may also be said of some of the efforts of high school math teachers in general.

**So what is the state of secondary geometry? It is the same as the state of elementary geometry!**

## Bibliography

Allen, Frank B., et al (1960). *Mathematics for High School: Geometry, Volumes I & II*. New Haven: Yale University Press.

Beutelispacher, Albrecht and Rosenbaum, Ute (1998). *Projective Geometry: From Foundations to Applications*. Cambridge, UK: Cambridge University Press. (ISBN: 0-521-48364-6)

Burger, W.F. (1992). A geometry curriculum for prospective elementary teachers based on the Van Hiele model of development. Pythagoras, 29, August, pp 9-17.

Burger, W.F. & Shaughnessy, J.M. (1986). Characterizing the Van Hiele levels of development in geometry. Journal for Research in Mathematics Education, 17, pp 31-48.

Crowley, Mary L. (1987). The Van Hiele model of the development of geometric thought. Learning and Teaching Geometry, K-12, 1987 Yearbook, pp 1-16. Arlington, VA: NCTM.

Devaney, Robert L. (1990). *Chaos, Fractals, and Dynamics: Computer Experiments in Mathematics*. Menlo Park, CA: Addison-Wesley Publishing Company. (ISBN: 0-201-23288-x)

De Villiers, Michael (1995). An alternative introduction to proof in dynamic geometry. MicroMath, Spring, pp 14-19.

De Villiers, Michael (1995b). The handling of geometry definitions in school textbooks. Pythagoras, 38, pp 3-4.

De Villiers, Michael (1996). The future of secondary school geometry. (Slightly adapted version of Plenary presented at the SOSI Geometry Imperfect Conference, October 2-4, 1996, UNISA, Pretoria.)

De Villiers, M.D. & Njisane, R.M. (1987). The development of geometric thinking among black high school pupils in KwaZulu. Proceedings of the 11<sup>th</sup> PME Conference, Montreal Vol 3, pp 117-123.

Durbin, John R. (1985). *Modern Algebra: An Introduction, 2E*. New York, NY: John Wiley and Sons. (ISBN: 0-471-88487-1)

Eves, Howard (1984). *An Introduction to the History of Mathematics, 5E*. Philadelphia, PA: Saunders Publishing. (ISBN: 0-030062064-3)

Gardner, Martin (2001). *The Collossal Book of Mathematics*. New York, NY: W.W. Norton and Company. (ISBN: 0-393-02023-1)

Govender, M. (1995). Pupils' proof-writing achievement in circle geometry. Unpublished B.Ed. dissertation, University of Durban-Westville.

Kay, David C. (2001). *College Geometry: A Discovery Approach, 2E*. Boston, MASS: Addison-Wesley Publishing Company. (ISBN: 0-321-04624-2)

Kilpatrick, J. & Wirzup, I. (1969). Soviet studies in the psychology of learning and teaching mathematics. Chicago, IL: University of Chicago.

Larson, Hostetler and Edwards (1999). *Calculus: Early Transcendental Functions, 2E*. Boston: Houghton-Mifflin Company. (ISBN: 0-395-93320-x)

Lee, Carl (1998). Some comments on geometry and models. [www.ms.uky.edu/~lee/ma502/notes5/node2.html](http://www.ms.uky.edu/~lee/ma502/notes5/node2.html)

Lott, Johnny W. editor (2001). *Navigating through Geometry in Grades 9-12*. Arlington, VA: NCTM. (ISBN: 0-87353-514-6)

Luthuli, D. (1996). Questions, reflection, and problem solving as sources of inquiry in Euclidean geometry. Pythagoras, 40, pp 17-27.

Malkevitch, Joseph, editor (1991). *Geometry's Future*. Arlington, VA: COMAP. (ISBN: 0912843217)

Malkevitch, Joseph (1999). New directions for geometry standards. [www.york.cuny.edu/~malk/CUPMGeometry/dimacs.geometry.html](http://www.york.cuny.edu/~malk/CUPMGeometry/dimacs.geometry.html)

Malkevitch, Joseph (2001). Geometry in utopia II.  
[www.york.cuny.edu/~malk/utopia.html](http://www.york.cuny.edu/~malk/utopia.html)

Malkevitch, Joseph (2002). Joseph Malkevitch's home page. [www.york.cuny.edu/~malk/](http://www.york.cuny.edu/~malk/)

Manning, Henry Parker (1914). *Geometry of Four Dimensions*. The Macmillan Company (Reprinted by Dover Publications, 1956).

Mlodinow, Leonard (2001). *Euclid's Window: The Story of Geometry from Parallel Lines to Hyperspace*. New York, NY: Free Press. (ISBN: 0-684-86523-8)

Moise, Edwin E. (1990). *Elementary Geometry from an Advanced Standpoint, 3E*. Reading, MASS: Addison-Wesley Publishing Company. (ISBN: 0-201-500867-2)

NCTM (2000). *Principles and Standards for School Mathematics*. Arlington, VA: NCTM. (ISBN: 0-87353-480-8)

Shaughnessy, J.M. & Burger, W.F. (1985). Spadework prior to deduction in geometry. Mathematics Teacher, 78, pp 419-28.

Smart, James (1998). *Modern Geometries, 2E*. Pacific Grove, CA: Brooks Cole Publishing Company. (ISBN: 0-534-35188-3)

Smith, R.R. (1940). Three major difficulties in the learning of demonstrative geometry. The Mathematics Teacher, 33, pp 99-134 & 150-178.

Thomas, David A. (2002). *Modern Geometry*. Pacific Grove, CA: Brooks Cole Publishing Company. (ISBN: 0-534-36550-7)

Thomas, David A. (1998). *Active Geometry*. Pacific Grove, CA: Brooks Cole Publishing Company. (ISBN: 0-534-34485)

Uzizkin, Zalman (1982). Van Hiele levels and achievement in secondary school geometry. Final report, Cognitive Development and Achievement in Secondary Schools Project. Chicago, IL: University of Chicago.

Wallace, Edward C. and West, Stephan E. (1998). *Roads to Geometry, 2E*. Upper Saddle River, NJ: Prentice Hall Publishing. (ISBN: 0-13-181652-7)

Wirzup (1976).