

Introduction

Focus

Spatial and geometric properties of the physical world were among the first mathematical ideas to emerge. Geometry still remains as a central focus of mathematics and is used to describe and analyze the world around us.

The focus of the following discussion is on the developments of geometry, the definition of geometry, recent issues of geometry education, and axiom sets for Euclidean geometry.

Discussion

Developments

To most lay people, geometry is the study of space and the shapes they witness in it. Their exposure to geometry is typically through classifying shapes that they learn in grade school and the "pseudo-axiomatic" geometry of high school. In high school most geometry concentrates on axiomatics and the proof of Euclidean theorems (via the two-column proof).

The word geometry comes from the Greek words "*geo*" meaning earth and "*metry*" meaning measure. The Greeks are credited with the first formally organized system of mathematics and mathematical thought. In particular, Euclid is given credit. However, many believe that mathematical reasoning originated in the Orient (lands east of Greece). Two examples include the work of the Sumerians (2100 BC) and the Babylonians (1600 BC). Most of their work concerned with the *how* of doing things, not the *why*.

When Ptolemy I had the University of Alexandria built, Euclid was placed as the head of the mathematics department (300 BC). Euclid produced several different texts. But, the one most remembered is the *Elements*. It is considered to be the second most studied text in the Western World. The *Bible* is the first.

The *Elements* consist of thirteen volumes. Volumes I, III, IV, VI, IX, and XII comprise the bulk of the material covered in a typical United States high school geometry class.

The *Elements* illustrate an axiomatic system of mathematics. The general structure of the system includes:

- undefined terms
- definitions
- axioms or postulates
- theorems
- logic

The logic used was developed by Aristotle (384-322 BC).

The system is said to be *consistent* if no axiom or theorem contradicts another. An axiom is said to be *independent* if it cannot be derived from the other axioms. Finally, an axiomatic system is *complete* if it is impossible to add a new independent axiom to it. The geometry described by Euclid's *Elements* is believed to be consistent and all axioms are believed to be independent. However, this was not always believed. In particular, several mathematicians believed Euclid's fifth axiom to be dependent upon the other axioms.

Kurt Gödel (1906-1978) demonstrated that rich systems such as arithmetic and Euclidean geometry are inheritantly incomplete.

This "believed" flaw and several others sparked the development of several other geometries and principles.

Modern geometries are considered to be the collection of geometries developed after 1800. The emergence of modern (abstract) algebra also had a major influence on geometry during this time. Felix Klein used recent findings to classify geometries in 1872. Central to his classification was the concepts of transformations.

Klein's definition of a geometry is a study of the properties of a set S that remain invariant when elements of the set S are subjected to the transformations of some transformation group Γ . The geometry is denoted by $G(S, \Gamma)$. The definition depends on a group as is defined in modern algebra. The definition below is the same except knowledge of groups is not necessary.

A geometry is a pair $G = (\Omega, I)$ where Ω is a set of objects (such as points and lines) and I is a relation on the set Ω that is both symmetric and reflexive. This is to say that if $x, y \in \Omega$ then:

- xIy implies yIx
- $xIx \forall x \in \Omega$

The studies done after 1800 included geometric transformations, extension of Euclidean geometry in the subject of triangles and circles, invention of projective geometry, and discoveries of non-Euclidean geometries. Riemannian geometry was instrumental in Einstein's establishing general relativity. Quantum geometry has come to the forefront during the last 20 years. It involves 10 spatial dimensions and 1 time dimension. This geometry shows promise in unifying general relativity and quantum mechanics, as well as providing the ultimate theory for describing the cosmos.

Reform

Geometry (usually considered Euclidean geometry) has itself undergone considerable change, as previously discussed. Several influential scholars have made contributions to this change. David Hilbert is probably the most important of these people during the twentieth century. His postulation of Euclidean geometry is widely used in geometry classrooms today. So too are G.D. Birkhoff's short list of axioms.

Another area that has seen great change recently is the instruction of geometry. Geometry education has experienced two major reforms during the last century. The School Mathematics Study Group (SMSG) and the Geometry's Future conference.

The SMSG was a project to revise the K-12 mathematics curriculum. The National Science Foundation (NSF) sponsored it and work began in 1958 under the leadership of Edward G. Begle of Yale University. The material developed for geometry still remains popular. The SMSG postulate set is also still used.

The Geometry's Future conference highlighted three main concepts.

- o New approaches in concepts are to be supported by reformed pedagogy.
- o Computer technology is to be incorporated into the course material.
- o Students are expected to understand more about mathematics and its connections, and be able to express these understandings verbally and in writing.

Specific goals from the proceedings of the conference include;

- use of an experimental and inductive point of view,
- applications of geometry,
- inclusion of recent developments,
- the use of figures and diagrams,
- the use of physical models,
- collaborative learning, and
- an emphasis on transformations.

The distinction between the mathematics of geometry and physical geometry is important. The meaning of and examples of mathematical proofs is very important.

These findings are also supported by the National Council of Teachers of Mathematics (NCTM) as expressed in both *Curriculum and Evaluation Standards for School Mathematics (1989)* and *Principles and Standards for School Mathematics (2000)*.

Standard 7 of *Curriculum and Evaluation Standards for School Mathematics (1989)* refers to the study of geometry synthetically, and Standard 8 refers to the study of geometry algebraically. The *Standards* also advocates for the inclusion of synthetic, coordinate and transformational geometry in high school geometry.

The *Principles and Standards for School Mathematics (2000)* maintains the importance of geometry in the P-12th grade mathematics curriculum. It includes geometry as one of its major bands of study. The focus of this emphasizes the students learning to --

- analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships;
- specify locations and describe spatial relationships using coordinate geometry and other representational systems;
- apply transformations and use symmetry to analyze mathematical situations;
- use visualization, spatial reasoning, and geometric modeling to solve problems.

State Standards

Several states (including Nebraska) have created or modified a set of mathematics standards in response to the No Child Left Behind Act (NCLBA). Nebraska had already started work

on its mathematics standards and assessments before NCLBA was enacted. The state chose to continue in the direction that it was headed and make the standards and assessments work with NCLBA.

NEBRASKA EIGHTH AND TWELTH GRADE GEOMETRY STANDARDS

8.4 GEOMETRY/SPATIAL CONCEPTS

8.4.1 By the end of eighth grade, students will identify, describe, compare, and classify two- and three dimensional geometric figures - plane figures like polygons and circles; solid figures like prisms, pyramids, cones, spheres, and cylinders; lines, line segments, rays, angles, parallel and perpendicular lines.

8.4.2 By the end of eighth grade, students will use geometric properties, the Pythagorean theorem, and the relationships of congruence, similarity, and symmetry.

8.4.3 By the end of eighth grade, students will use formulas to solve problems involving perimeter and area of a square, rectangle, parallelogram, trapezoid and triangle, as well as the area and circumference of circles.

8.4.4 By the end of eighth grade, students will solve problems given formulas for volume and surface area of rectangular prisms, cylinders, and cones.

8.4.5 By the end of eighth grade, students will apply transformations to two- and three-dimensional geometric figures.

Example indicator:

- Draw geometric figures using translations or slides, rotations or turns, reflections or flips, and scale.

8.4.6 By the end of eighth grade, students will use geometric terms and representations to describe the physical world.

12.4 GEOMETRY/SPATIAL CONCEPT

12.4.1 By the end of twelfth grade, students will calculate perimeter and area of two-dimensional shapes and surface area and volume of three-dimensional shapes.

12.4.2 By the end of twelfth grade, students will create geometric models to describe the physical world.

Example indicators:

- Create perspective drawing.
- Create scale models.

12.4.3 By the end of twelfth grade, students will evaluate characteristics and properties of two- and three-dimensional geometric shapes.

Example indicators:

- Classify and compare attributes of two- and three-dimensional shapes.
- Classify shapes in terms of congruence and similarity and apply these relationships.
- Determine the effects of changing dimensions on perimeter, area, and volume.
- Investigate and deduce geometric properties using transformations such as translations, rotations, and reflections.

12.4.4 By the end of twelfth grade, students will apply coordinate geometry to locate and describe objects algebraically.

Example indicators:

- Graph a geometric shape and determine the slope of the sides.
- Identify the missing vertices of a polygon.

12.4.5 By the end of twelfth grade, students will apply right triangle trigonometry to find length and angle measures.

12.4.6 By the end of twelfth grade, students will apply geometric properties to solve problems.

Example indicator:

- Find missing angles and lengths of geometric shapes using geometric properties. (Properties may include but are not limited to similarity, parallel and line-transversal).

12.4.7 By the end of twelfth grade, students will apply deductive reasoning to arrive at a conclusion.

Example indicators:

- Justify steps when solving an algebraic equation using properties of real numbers.
- Use logic statements, paragraph proof, two-column proof, or algebraic proof to arrive at a conclusion.

Axiom Sets

EUCLID'S FIRST FIVE AXIOMS

Axiom 1. A line may be drawn between any two points.

Axiom 2. A segment of any length may be constructed in any line.

Axiom 3. A circle of any radius and center may be drawn.

Axiom 4. All right angles are congruent.

Axiom 5. Through a point A not on line m (such that A and m are on the same plane), exactly one line may be drawn parallel to m .

SMSG POSTULATE SET

POSTULATE 1. Given any two distinct points there is exactly one line that contains them.

POSTULATE 2. (The Distance Postulate) To every pair of distinct points there corresponds a unique positive number. This number is called the distance between the two points.

POSTULATE 3. (The Ruler Postulate) The points of a line can be placed in a correspondence with the real numbers such that:

1. To every point of the line there corresponds exactly one real number,
2. To every real number there corresponds exactly one point of the line, and
3. The distance between two distinct points is the absolute value of the difference of the corresponding real numbers.

POSTULATE 4. (The Ruler Placement Postulate) Given two points P and Q of a line, the coordinate system can be chosen in such a way that the coordinate of P is zero and the coordinate of Q is positive.

POSTULATE 5.

- (a) Every plane contains at least three non-collinear points.
- (b) Space contains at least four non-coplanar points.

POSTULATE 6. If two points lie in a plane, then the line containing these points lies in the same plane.

POSTULATE 7. Any three points lie in at least one plane, and any three non-collinear points lie in exactly one plane.

POSTULATE 8. If two planes intersect, then that intersection is a line.

POSTULATE 9. (The Plane Separation Postulate) Given a line and a plane containing it, the points of the plane that do not lie on the line form two sets such that:

1. Each of the sets is convex and
2. If P is in one set and Q is in the other, then segment \overline{PQ} intersects the line.

POSTULATE 10. (The Space Separation Postulate) The points of space that do not lie in a given plane form two sets such that:

1. Each of the sets is convex and
2. If P is in one set and Q is in the other, then segment \overline{PQ} intersects the line.

POSTULATE 11. (The Angle Measurement Postulate) To every angle there corresponds a real number between 0° and 180° .

- POSTULATE 12.** (The Angle Construction Postulate) Let \overrightarrow{AB} be a ray on the edge of the half-plane H . For every r between 0° and 180° there is exactly one ray \overrightarrow{AP} , with P in H such that $m\angle PAB = r$.
- POSTULATE 13.** (The Angle Addition Postulate) If D is a point in the interior of $\angle BAC$, then $m\angle BAC = m\angle BAD + m\angle DAC$.
- POSTULATE 14.** (The Supplement Postulate) If two angles form a linear pair, then they are supplementary.
- POSTULATE 15.** (The SAS Postulate) Given a one-to-one correspondence between two triangles (or between a triangle and itself). If two sides and the included angle of the first triangle are congruent to the corresponding parts of the second triangle, then the correspondence is congruence.
- POSTULATE 16.** (The Parallel Postulate) Through a given external point there is at most one line parallel to a given line.
- POSTULATE 17.** To every polygonal region there corresponds a unique positive real number called its area.
- POSTULATE 18.** If two triangles are congruent, then the triangular regions have the same area.
- POSTULATE 19.** Suppose that the region R is the union of two regions R_1 and R_2 . If R_1 and R_2 intersect at most in a finite number of segments and points, then the area of R is the sum of the areas of R_1 and R_2 .
- POSTULATE 20.** The area of a rectangle is the product of the length of its base and the length of its altitude.
- POSTULATE 21.** The volume of a rectangular parallelepiped is equal to the product of the length of its altitude and the area of its base.
- POSTULATE 22.** (Cavalieri's Principle) Given two solids and a plane. If for every plane that intersects the solids and is parallel to the given plane the two intersections determine regions that have the same area, then the two solids have the same volume.

HILBERT'S AXIOM SET

I. Axioms of Incidence (Connection)

- Postulate I.1** Through any two distinct points A , B , there is a line, m .
- Postulate I.2** Through any two distinct points A , B , there is not more than one line, m .
- Postulate I.3** There exists at least two points on any given line. There exists at least three points that do not lie on a given line.

Postulate I.4 For a set of three points $\{A, B, C\}$ that do not lie on the same line, there exists a plane P that contains each of the points in the set. For every plane there exists at least one point which it contains.

Postulate I.5 For a set of three points $\{A, B, C\}$ that do not all lie on the same line, there exists only one plane P that contains each of the points in the set.

Postulate I.6 If two points A, B of a line, m , lie in a plane, P , then every point in m lies in P .

Postulate I.7 If two planes P, Q have a point A in common, then they have at least one other point, B , in common.

Postulate 1.8 There exist at least four points which do not lie in a plane.

II. Axioms of Order

Postulate II.1 If a point B lies between points A and C , then the points $\{A, B, C\}$ are three distinct points on the same line and B lies between C and A .

Postulate II.2 Given two points A, C , a point B exists on the line \overleftrightarrow{AC} such that C lies between A and B .

Postulate II.3 Given any three points $\{A, B, C\}$ of a line, one and only one of the points is between the other two.

Postulate II.4 Given three points $\{A, B, C\}$ that do not lie on a line and given a line, m , that lies in the plane ABC which does not intersect any of the points A, B, C : if the line m passes through a point of the segment \overline{AB} , it also passes through a point in the segment \overline{AC} or through a point in the segment \overline{BC} .

III. Axioms of Congruence

Postulate III.1 Given two points A, B on a line m and given a point A' on a or another line m' , there exists a point B' on a side of the line m' such that \overline{AB} and $\overline{A'B'}$ are congruent.

Postulate III.2 Given segments $\overline{A'B'}$ and $\overline{A''B''}$ such that both are congruent to the same segment \overline{AB} , then $\overline{A'B'} \cong \overline{A''B''}$

Postulate III.3 Given a line m with segments \overline{AB} and \overline{BC} such that the point B is the only intersection of the two points and on the same line or a line m' with segments $\overline{A'B'}$ and $\overline{B'C'}$ such that the point B' is the only intersection: if $\overline{AB} \cong \overline{A'B'}$ and $\overline{BC} \cong \overline{B'C'}$ then $\overline{AC} \cong \overline{A'C'}$.

Postulate III.4 If $\angle ABC$ is an angle and $\overrightarrow{B'C'}$ is a ray, then there is one and only one ray $\overrightarrow{B'A'}$ on each side of the line $\overleftrightarrow{B'C'}$ such that $\angle A'B'C' \cong \angle ABC$.

Corollary: Every angle is congruent to itself.

Postulate III.5 (Side-Angle-Side Axiom) Given two triangles $\triangle ABC$ and $\triangle A'B'C'$ with congruences such that $\overline{AB} \cong \overline{A'B'}$, $\overline{AC} \cong \overline{A'C'}$ and $\angle BAC \cong \angle B'A'C'$ then $\triangle ABC \cong \triangle A'B'C'$.

IV. Axiom of Parallels

Postulate IV.1 Given a line m and a point A not on m , there is at most one line in the plane that contains m and A that passes through A and does not intersect m .

V. Axioms of Continuity

Postulate V.1 (Archimedes Axiom) Given segments \overline{AB} and \overline{CD} , there exists a number n such that n copies of \overline{CD} constructed contiguously from A along the ray \overrightarrow{AB} will pass beyond the point B .

Postulate V.2 (Line Completeness) There exists no extension of a set of points on a line with order and congruence relations that would preserve the relations existing among the original elements as well as preserving line order and congruence, i.e., Axioms I-III and V.1.